Riesz and Szegö type factorizations for noncommutative Hardy spaces

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Abstract

Let A be a finite subdiagonal algebra in Arveson's sense. Let $H^p(\mathsf{A})$ be the associated noncommutative Hardy spaces, $0 < p \leq \infty$. We extend to the case of all positive indices most recent results about these spaces, which include notably the Riesz, Szegö and inner-outer type factorizations. One new tool of the paper is the contractivity of the underlying conditional expectation on $H^p(\mathsf{A})$ for $p < 1$.

1 Introduction

This paper deals with the Riesz and Szegö type factorizations for noncommutative Hardy spaces associated with a finite subdiagonal algebra in Arveson's sense [1]. Let M be a finite von Neumann algebra equipped with a normal faithful tracial state τ. Let D be a von Neumann subalgebra of M, and let $\Phi : \mathsf{M} \to \mathsf{D}$ be the unique normal faithful conditional expectation such that $\tau \circ \Phi = \tau$. A *finite subdiagonal algebra* of M with respect to Φ (or D) is a w*-closed subalgebra A of M satisfying the following conditions

i) $\mathsf{A} + \mathsf{A}^*$ is w*-dense in M;

ii) Φ is multiplicative on A, i.e., $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathsf{A}$;

iii) $\mathsf{A} \cap \mathsf{A}^* = \mathsf{D}$.

We should call the reader's attention to fact that A^* denotes in this paper the family of the adjoints of the elements of A, i.e., $\mathsf{A}^* = \{a^* \ : \ a \in \mathsf{A}\}$. The algebra D is called the *diagonal* of A. It is proved by Exel [6] that a finite subdiagonal algebra A is automatically *maximal* in the sense that if B is another subdiagonal algebra with respect to Φ containing A, then $\mathsf{B} = \mathsf{A}$. This maximality yields the following useful characterization of A:

$$\mathsf{A} = \{x \in \mathsf{M} \ : \ \tau(xa) = 0, \ \forall\, a \in \mathsf{A}_0\}, \tag{1.1}$$

where $\mathsf{A}_0 = \mathsf{A} \cap \ker \Phi$ (see [1]).

Given $0 < p \leq \infty$ we denote by $L^p(\mathsf{M})$ the usual noncommutative L^p-space associated with (M, τ). Recall that $L^\infty(\mathsf{M}) = \mathsf{M}$, equipped with the operator norm. The norm of $L^p(\mathsf{M})$ will be denoted by $\| \cdot \|_p$. For $p < \infty$ we define $H^p(\mathsf{A})$ to be the closure of A in $L^p(\mathsf{M})$, and for $p = \infty$ we simply set $H^\infty(\mathsf{A}) = \mathsf{A}$ for convenience. These are the so-called Hardy spaces associated with

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A. They are noncommutative extensions of the classical Hardy spaces on the torus $\mathbb{T}$. On the other hand, the theory of matrix-valued analytic functions provides an important noncommutative example. We refer to [1] and [14] for more examples. We will use the following standard notation in the theory: If S is a subset of $L^p(\mathsf{M})$, $[S]_p$ will denote the closure of S in $L^p(\mathsf{M})$ (with respect to the w*-topology in the case of $p = \infty$). Thus $H^p(\mathsf{A}) = [\mathsf{A}]_p$. Formula (1.1) admits the following $H^p(\mathsf{A})$ analogue proved by Saito [15]:

$$H^p(\mathsf{A}) = \{x \in L^p(\mathsf{M}) \ : \ \tau(xa) = 0, \ \forall\, a \in \mathsf{A}_0\}, \quad 1 \le p < \infty. \tag{1.2}$$

Moreover,

$$H^p(\mathsf{A}) \cap L^q(\mathsf{M}) = H^q(\mathsf{A}), \quad 1 \le p < q \le \infty. \tag{1.3}$$

These noncommutative Hardy spaces have received a lot of attention since Arveson's pioneer work. We refer the reader notably to the recent work by Marsalli/West [13] and a series of newly finished papers by Blecher/Labuschagne [2, 3, 4], whereas more references on previous works can be found in the survey paper [14]. Most results on the classical Hardy spaces on the torus have been established in this noncommutative setting. Here we mention only two of them directly related with the objective of this paper. The first one is the Szegö factorization theorem. Already in the fundamental work [1], Arveson proved the following factorization theorem: For any invertible $x \in \mathsf{M}$ there exist a unitary $u \in \mathsf{M}$ and $a \in \mathsf{A}$ such that $x = ua$ and $a^{-1} \in \mathsf{A}$. This theorem is a base of all subsequent works on noncommutative Hardy spaces. It has been largely improved and extended. The most general form up to date was newly obtained by Blecher and Labuschagne [2]: Given $x \in L^p(\mathsf{M})$ with $1 \le p \le \infty$ such that $\Delta(x) > 0$ there exists $h \in H^p(\mathsf{M})$ such that $|x| = |h|$. Moreover, h is *outer* in the sense that $[h\mathsf{A}]_p = H^p(\mathsf{M})$. Here $\Delta(x)$ denotes the Fuglede-Kadison determinant of x (see section 2 below for the definition), and $|x| = (x^*x)^{1/2}$ denotes the absolute value of x. We should emphasize that this result is the (almost) perfect analogue of the classical Szegö theorem which asserts that given a positive measurable function w on the torus there exists an outer function φ such that $w = |\varphi|$ iff $\log w$ is integrable.

The second result we wish to mention concerns the Riesz factorization, which asserts that $H^p(\mathsf{A}) = H^q(\mathsf{A}) \cdot H^r(\mathsf{A})$ for any $1 \le p, q, r \le \infty$ such that $1/p = 1/q + 1/r$. More precisely, given $x \in H^p(\mathsf{A})$ and $\varepsilon > 0$ there exist $y \in H^q(\mathsf{A})$ and $z \in H^r(\mathsf{A})$ such that

$$x = yz \quad \text{and} \quad \|y\|_q \, \|z\|_r \le \|x\|_p + \varepsilon.$$

This result is proved in [15] for $p = q = 2$, in [13] for $r = 1$ and independently in [12] and in [14] for the general case as above.

Recall that in the case of the classical Hardy spaces the preceding theorems hold for all positive indices. The problem of extending these results to the case of indices less than one was left unsolved in these works. (We mentioned this problem for the Riesz factorization explicitly in [14], see the remark following Theorem 8.3 there). The main purpose of the present paper is to solve the problem above. As a byproduct, we also extend all results on outer operators in [2] to indices less than one.

A major obstacle to the solution of the previous problem is the use of duality, often in a crucial way, in the literature on noncommutative Hardy spaces. For instance, duality plays an important role in proving formulas (1.2) and (1.3), which are key ingredients for the Riesz factorization in [13]. In a similar fashion, we will see that their extensions to indices less than one will be essential for our proof of the Riesz factorization for all positive indices.

Our key new tool is the contractivity of the conditional expectation Φ on A with respect to $\|\cdot\|_p$ for $0 < p < 1$. Consequently, Φ extends to a contractive projection from $H^p(\mathsf{A})$ onto $L^p(\mathsf{D})$. This result is of independent interest and proved in section 2.

Section 3 is devoted to the Szegö and Riesz type factorizations. In particular, we extend to all positive indices Marsalli/West's theorem quoted previously. Section 4 contains some results on outer operators, notably those in $H^p(\mathsf{A})$ for $p < 1$. This section can be considered as a complement

to the recent work [2]. The last section is devoted to a noncommutative Szegö formula, which was obtained in [2] with the additional assumption that $\dim \mathsf{D} < \infty$.

We will keep all previous notations throughout the paper. In particular, A will always denote a finite subdiagonal algebra of (M, τ) with diagonal D.

2 Contractivity of Φ on $H^p(\mathsf{A})$ for $p < 1$

It is well-known that Φ extends to a contractive projection from $L^p(\mathsf{M})$ onto $L^p(\mathsf{D})$ for every $1 \leq p \leq \infty$. In general, Φ cannot be, of course, continuously extended to $L^p(\mathsf{M})$ for $p < 1$. Surprisingly, Φ does extend to a contractive projection on $H^p(\mathsf{A})$.

Theorem 2.1 *Let $0 < p < 1$. Then*

$$\forall\, a \in \mathsf{A} \quad \|\Phi(a)\|_p \leq \|a\|_p\,. \tag{2.1}$$

Consequently, Φ extends to a contractive projection from $H^p(\mathsf{A})$ onto $L^p(\mathsf{D})$. The extension will be denoted still by Φ.

Inequality (2.1) is proved by Labuschagne [11] for $p = 2^{-n}$ and for operators a in A which are invertible with inverses in A too. Labuschagne's proof is a very elegant and simple argument by induction. It can be adapted to our general situation.

Proof. Since $\{k2^{-n} \ :\ k, n \in \mathbb{N},\ k \geq 1\}$ is dense in $(0,1)$, it suffices to prove (2.1) for $p = k2^{-n}$. Thus we must show

$$\forall\, a \in \mathsf{A} \quad \tau\big(|\Phi(a)|^{k2^{-n}}\big) \leq \tau\big(|a|^{k2^{-n}}\big). \tag{2.2}$$

This inequality holds for $n = 0$ because of the contractivity of Φ on $L^k(\mathsf{M})$. Now suppose its validity for some k and n. We will prove the same inequality with $n+1$ instead of n. To this end fix $a \in \mathsf{A}$ and $\varepsilon > 0$. Define, by induction, a sequence (x_m) by

$$x_1 = (|a| + \varepsilon)^{k2^{-n}} \quad \text{and} \quad x_{m+1} = \frac{1}{2}\big[x_m + (|a| + \varepsilon)^{k2^{-n}}\, x_m^{-1}\big].$$

Observe that all x_m belong to the commutative C*-subalgebra generated by $|a|$. Then it is an easy exercise to show that the sequence (x_m) is nonincreasing and converges to $(|a|+\varepsilon)^{k2^{-n-1}}$ uniformly (see [11]). We also have

$$\begin{aligned}
\tau(x_{m+1}) &= \frac{1}{2}\big[\tau(x_m) + \tau\big(x_m^{-1/2}(|a| + \varepsilon)^{k2^{-n}}\, x_m^{-1/2}\big)\big]\\
&\geq \frac{1}{2}\big[\tau(x_m) + \tau\big(x_m^{-1/2}|a|^{k2^{-n}}\, x_m^{-1/2}\big)\big]\\
&= \frac{1}{2}\big[\tau(x_m) + \tau\big(|a|^{k2^{-n}}\, x_m^{-1}\big)\big].
\end{aligned}$$

Now applying Arveson's factorization theorem to each x_m, we find an invertible $b_m \in \mathsf{A}$ with $b_m^{-1} \in \mathsf{A}$ such that

$$|b_m| = x_m^{2^n/k}\,.$$

Let $p = k2^{-n}$. Then

$$\begin{aligned}
\big\|ab_m^{-1}\big\|_p &= \big\||a|\, b_m^{-1}\big\|_p = \big\||a|\, |(b_m^{-1})^*|\big\|_p\\
&= \big\||a|\, |b_m|^{-1}\big\|_p = \big(\tau(|a|^p|b_m|^{-p})\big)^{1/p}\\
&= \big(\tau(|a|^p\, x_m^{-1})\big)^{1/p},
\end{aligned}$$

where we have used the commutation between $|a|$ and $|b_m|$ for the next to the last equality. Therefore, by the induction hypothesis and the multiplicativity of Φ on A

$$\begin{aligned}\tau(x_{m+1}) &\geq \frac{1}{2}\big[\tau\big(|b_m|^{k2^{-n}}\big)+\tau\big(|ab_m^{-1}|^{k2^{-n}}\big)\big]\\ &\geq \frac{1}{2}\big[\tau\big(|\Phi(b_m)|^{k2^{-n}}\big)+\tau\big(|\Phi(a)\Phi(b_m)^{-1}|^{k2^{-n}}\big)\big].\end{aligned}$$

However, by the Hölder inequality

$$\big(\tau\big(|\Phi(a)|^{k2^{-n-1}}\big)\big)^2 \leq \tau\big(|\Phi(a)\Phi(b_m)^{-1}|^{k2^{-n}}\big)\,\tau\big(|\Phi(b_m)|^{k2^{-n}}\big).$$

It thus follows that

$$\begin{aligned}\tau(x_{m+1}) &\geq \frac{1}{2}\big[\tau\big(|\Phi(b_m)|^{k2^{-n}}\big)+\big(\tau\big(|\Phi(a)|^{k2^{-n-1}}\big)\big)^2\big(\tau\big(|\Phi(b_m)|^{k2^{-n}}\big)\big)^{-1}\big]\\ &\geq \tau\big(|\Phi(a)|^{k2^{-n-1}}\big).\end{aligned}$$

Recalling that $x_m \to (|a|+\varepsilon)^{k2^{-n-1}}$ as $m\to\infty$, we deduce

$$\tau\big((|a|+\varepsilon)^{k2^{-n-1}}\big) \geq \tau\big(|\Phi(a)|^{k2^{-n-1}}\big).$$

Letting $\varepsilon \to 0$ we obtain inequality (2.2) at the $(n+1)$-th step. $\square$

Corollary 2.2 *Φ is multiplicative on Hardy spaces. More precisely, $\Phi(ab) = \Phi(a)\Phi(b)$ for $a \in H^p(\mathsf{A})$ and $b \in H^q(\mathsf{A})$ with $0 < p, q \leq \infty$.*

Proof. Note that $ab \in H^r(\mathsf{A})$ for any $a \in H^p(\mathsf{A})$ and $b \in H^q(\mathsf{A})$, where r is determined by $1/r = 1/p + 1/q$. Thus $\Phi(ab)$ is well defined. Then the corollary follows immediately from the multiplicativity of Φ on A and Theorem 2.1. $\square$

The following is the extension to the case $p < 1$ of Arveson-Labuschagne's Jensen inequality (cf. [1, 11]). Recall that the Fuglede-Kadison determinant $\Delta(x)$ of an operator $x \in L^p(\mathsf{M})$ $(0 < p \leq \infty)$ can be defined by

$$\Delta(x) = \exp\big(\tau(\log|x|)\big) = \exp\Big(\int_0^\infty \log t\, d\nu_{|x|}(t)\Big),$$

where $d\nu_{|x|}$ denotes the probability measure on $\mathbb{R}_+$ which is obtained by composing the spectral measure of $|x|$ with the trace τ. It is easy to check that

$$\Delta(x) = \lim_{p\to 0}\|x\|_p\,.$$

As the usual determinant of matrices, Δ is also multiplicative: $\Delta(xy) = \Delta(x)\Delta(x)$. We refer the reader for information on determinant to [7, 1] in the case of bounded operators, and to [5, 9] for unbounded operators.

Corollary 2.3 *For any $0 < p \leq \infty$ and $x \in H^p(\mathsf{A})$ we have $\Delta(\Phi(x)) \leq \Delta(x)$.*

Proof. Let $x \in H^p(\mathsf{A})$. Then $x \in H^q(\mathsf{A})$ too for $q \leq p$. Thus by Theorem 2.1

$$\|\Phi(x)\|_q \leq \|x\|_q\,.$$

Letting $q \to 0$ yields $\Delta(\Phi(x)) \leq \Delta(x)$. $\square$

3 Szegö and Riesz factorizations

The following result is a Szegö type factorization theorem. It is stated in [14] without proof (see the remark following Theorem 8.1 there). We take this opportunity to provide a proof. It is an improvement of the previous factorization theorems of Arveson [1] and Saito [15]. As already quoted in the introduction, Blecher and Labuschagne newly obtained a Szegö factorization for any $w \in L^p(\mathsf{M})$ with $1 \leq p \leq \infty$ such that $\Delta(w) > 0$ (see the next section for more details). Note that the property that $h^{-1} \in H^q(\mathsf{A})$ whenever $w^{-1} \in L^q(\mathsf{M})$ will be important for our proof of the Riesz factorization below. Let us also point out that although not in full generality, this result has hitherto been strong enough for applications in the literature. See Theorem 4.8 below for an improvement.

Theorem 3.1 *Let $0 < p, q \leq \infty$. Let $w \in L^p(\mathsf{M})$ be an invertible operator such that $w^{-1} \in L^q(\mathsf{M})$. Then there exist a unitary $u \in \mathsf{M}$ and $h \in H^p(\mathsf{A})$ such that $w = uh$ and $h^{-1} \in H^q(\mathsf{A})$.*

Proof. We first consider the case $p = q = 2$. The proof of this special case is modelled on Arveson's original proof of his Szegö factorization theorem (see also [15]). Let x be the orthogonal projection of w in $[w\mathsf{A}_0]_2$, and set $y = w - x$. Thus $y \perp [w\mathsf{A}_0]_2$; whence $y \perp [y\mathsf{A}_0]_2$. It follows that

$$\forall\, a \in \mathsf{A}_0 \quad \tau(y^*ya) = 0.$$

Hence by (1.2), $y^*y \in H^1(\mathsf{A}) = [\mathsf{A}]_1$, and $y^*y \in [\mathsf{A}^*]_1$ too. On the other hand, it is easy to see that $[\mathsf{A}]_1 \cap [\mathsf{A}^*]_1 = L^1(\mathsf{D})$. Indeed, if $a \in [\mathsf{A}]_1 \cap [\mathsf{A}^*]_1$, then $\tau(ab) = 0$ for any $b \in \mathsf{A}_0 + \mathsf{A}_0^*$; so $\tau(ab) = \tau(\Phi(a)b)$ for any $b \in \mathsf{A}+\mathsf{A}^*$. It follows that $a = \Phi(a) \in L^1(\mathsf{D})$. Consequently, $y^*y \in L^1(\mathsf{D})$, so $|y| \in L^2(\mathsf{D})$.

Regarding M as a von Neumann algebra acting on $L^2(\mathsf{M})$ by left multiplication, we claim that y is cyclic for M. This is equivalent to showing that y is separating for the commutant of M. However, this commutant coincides with the algebra of all right multiplications on $L^2(\mathsf{M})$ by the elements of M. Thus we are reduced to prove that if $z \in \mathsf{M}$ is such that $yz = 0$, then $z = 0$. We have:

$$0 = \tau(z^*y^*yz) = \tau(|y|^2\,|z^*|^2) = \tau(|y|^2\,\Phi(|z^*|^2)) = \|yd\|_2^2\,,$$

where $d = \Phi(|z^*|^2)^{1/2} \in \mathsf{D}$; whence $yd = 0$. Choose a sequence $(a_n) \subset \mathsf{A}_0$ such that

$$x = \lim wa_n. \tag{3.1}$$

Then (recalling that $w^{-1} \in L^2(\mathsf{M})$)

$$0 = \tau(w^{-1}yd) = \lim_n \tau(w^{-1}(w - wa_n)d) = \tau(d) - \lim_n \tau(a_nd) = \tau(d).$$

It follows that $d = 0$, so by virtue of the faithfulness of Φ, $z = 0$ too. This yields our claim. Therefore, $[\mathsf{M}y]_2 = L^2(\mathsf{M})$. It turns out that the right support of y is 1. Since M is finite, the left support of y is also equal to 1, so y is of full support. Consequently, $[y\mathsf{M}]_2 = L^2(\mathsf{M})$ too.

Let $y = u|y|$ be the polar decomposition of y. Then u is a unitary in M. Let $h = u^*w$. We are going to prove that $h \in H^2(\mathsf{A})$. To this end we first note the following orthogonal decomposition of $L^2(\mathsf{M})$:

$$L^2(\mathsf{M}) = [y\mathsf{A}_0]_2 \oplus [y\mathsf{D}]_2 \oplus [y\mathsf{A}_0^*]_2\,. \tag{3.2}$$

Indeed, for any $a \in \mathsf{A}$ and $b \in \mathsf{A}_0$ we have

$$\langle ya,\ yb^*\rangle = \tau(by^*ya) = \tau(|y|^2ab) = 0;$$

so $[y\mathsf{A}_0]_2 \oplus [y\mathsf{D}]_2 \oplus [y\mathsf{A}_0^*]_2$ is really an orthogonal sum. On the other hand, by the previous paragraph, we see that

$$L^2(\mathsf{M}) = [y\mathsf{M}]_2 \subset [y\mathsf{A}_0]_2 \oplus [y\mathsf{D}]_2 \oplus [y\mathsf{A}_0^*]_2\,.$$

Therefore, decomposition (3.2) follows. Applying u^* to both sides of (3.2), we deduce

$$\begin{aligned} L^2(\mathsf{M}) &= [u^*y\mathsf{A}_0]_2 \oplus [u^*y\mathsf{D}]_2 \oplus [u^*y\mathsf{A}_0^*]_2 \\ &= [|y|\mathsf{A}_0]_2 \oplus [|y|\mathsf{D}]_2 \oplus [|y|\mathsf{A}_0^*]_2 \,. \end{aligned}$$

Since $|y| \in L^2(\mathsf{D})$, $[|y|\mathsf{A}_0]_2 \subset [A_0]_2$, and similarly for the two other terms on the right. Therefore,

$$\begin{aligned} L^2(\mathsf{M}) &= [|y|\mathsf{A}_0]_2 \oplus [|y|\mathsf{D}]_2 \oplus [|y|\mathsf{A}_0^*]_2 \\ &\subset [\mathsf{A}_0]_2 \oplus [\mathsf{D}]_2 \oplus [\mathsf{A}_0^*]_2 = L^2(\mathsf{M}) \,. \end{aligned}$$

Hence

$$[|y|\mathsf{A}_0]_2 = [\mathsf{A}_0]_2, \quad [|y|\mathsf{D}]_2 = [\mathsf{D}]_2, \quad [|y|\mathsf{A}_0^*]_2 = [\mathsf{A}_0^*]_2 \,. \tag{3.3}$$

Passing to adjoints, we also have

$$[\mathsf{A}_0|y|]_2 = [\mathsf{A}_0]_2, \quad [\mathsf{D}|y|]_2 = [\mathsf{D}]_2, \quad [\mathsf{A}_0^*|y|]_2 = [\mathsf{A}_0^*]_2 \,.$$

Now it is easy to show that $h = u^*w \in H^2(\mathsf{A})$. Indeed, since $y \perp [w\mathsf{A}_0]$, $\tau(y^*wa) = 0$ for all $a \in \mathsf{A}_0$; so $\tau(a|y|u^*w) = 0$. However, $[\mathsf{A}_0|y|]_2 = [\mathsf{A}_0]_2$. Thus

$$\forall\, a \in H^2_0(\mathsf{A}) \quad \tau(ah) = 0.$$

Hence by (1.2), $h \in H^2(\mathsf{A})$, as desired.

It remains to show that $h^{-1} \in H^2(\mathsf{A})$. To this end we first observe that $\Phi(h)\Phi(h^{-1}) = 1$. Indeed, given $d \in \mathsf{D}$ we have, by (3.1)

$$\begin{aligned} \tau\big(\Phi(h)\Phi(h^{-1})|y|d\big) &= \tau\big(h^{-1}|y|d\Phi(h)\big) = \tau\big(w^{-1}u|y|d\Phi(h)\big) \\ &= \lim_n \tau\big(w^{-1}(w - wa_n)d\Phi(h)\big) = \tau\big(d\Phi(h)\big) \\ &= \tau(hd) = \tau(u^*wd) = \tau(u^*yd) = \tau(|y|d), \end{aligned}$$

where we have used the fact that

$$\tau(u^*xd) = \lim_n \tau(u^*wa_nd) = \lim_n \tau(ha_nd) = 0.$$

Since $[|y|\mathsf{D}]_2 = L^2(\mathsf{D})$, we deduce our observation. Therefore, $\Phi(h)$ is invertible and its inverse is $\Phi(h^{-1})$. On the other hand, by (3.1)

$$\Phi(h) = \lim_n \Phi(u^*(y + wa_n)) = \Phi(|y|) + \lim_n \Phi(ha_n) = u^*y.$$

Hence,

$$u = y\Phi(h)^{-1} = y\Phi(h^{-1}).$$

Now let $a \in \mathsf{A}_0$. Then

$$\tau(h^{-1}a) = \tau(w^{-1}ua) = \tau\big(w^{-1}y\Phi(h^{-1})a\big) = \lim_n \tau\big(w^{-1}(w - wa_n)\Phi(h^{-1})a\big) = 0.$$

It follows that $h^{-1} \in H^2(\mathsf{A})$. Therefore, we are done in the case $p = q = 2$.

The general case can be easily reduced to this special one. Indeed, if $p \geq 2$ and $q \geq 2$, then given $w \in L^p(\mathsf{M})$ with $w^{-1} \in L^q(\mathsf{M})$, we can apply the preceding part and then find a unitary $u \in \mathsf{M}$ and $h \in H^2(\mathsf{A})$ such that $w = uh$ and $h^{-1} \in H^2(\mathsf{A})$. Then $h = u^*w \in L^p(\mathsf{M})$, so $w \in H^2(\mathsf{A}) \cap L^p(\mathsf{M}) = H^p(\mathsf{A})$ by (1.3). Similarly, $h^{-1} \in H^q(\mathsf{A})$.

Suppose $\min(p,\, q) < 2$. Choose an integer n such that $\min(np,\, nq) \geq 2$. Let $w = v|w|$ be the polar decomposition of w. Note that $v \in \mathsf{M}$ is a unitary. Write

$$w = v|w|^{1/n}\,|w|^{1/n}\,\cdots\,|w|^{1/n} = w_1w_2\cdots w_n,$$

where $w_1 = v|w|^{1/n}$ and $w_k = |w|^{1/n}$ for $2 \le k \le n$. Since $w_k \in L^{np}(\mathsf{M})$ and $w_k^{-1} \in L^{nq}(\mathsf{M})$, by what is already proved we have a factorization

$$w_n = u_n h_n$$

with $u_n \in \mathsf{M}$ a unitary, $h_n \in H^{np}(\mathsf{A})$ such that $h_n^{-1} \in H^{nq}(\mathsf{A})$. Repeating this argument, we again get a same factorization for $w_{n-1}u_n$:

$$w_{n-1}u_n = u_{n-1}h_{n-1}\,;$$

and then for $w_{n-2}u_{n-1}$, and so on. In this way, we obtain a factorization:

$$w = uh_1 \cdots h_n,$$

where $u \in \mathsf{M}$ is a unitary, $h_k \in H^{np}(\mathsf{A})$ such that $h_k^{-1} \in H^{nq}(\mathsf{A})$. Setting $h = h_1 \cdots h_n$, we then see that $w = uh$ is the desired factorization. Hence the proof of the theorem is complete. $\square$

Remark 3.2 Let $w \in L^2(\mathsf{M})$ be an invertible operator such that $w^{-1} \in L^2(\mathsf{M})$. Let $w = uh$ be the factorization in Theorem 3.1. The preceding proof shows that $[h\mathsf{A}]_2 = H^2(\mathsf{A})$. Indeed, it is clear that $[y\mathsf{A}]_2 \subset [w\mathsf{A}]_2$. Using decomposition (3.2), we get

$$[w\mathsf{A}]_2 \ominus [y\mathsf{A}]_2 = [w\mathsf{A}]_2 \cap [y\mathsf{A}_0^*]_2\,.$$

Now for any $a \in \mathsf{A}$ and $b \in \mathsf{A}_0$,

$$\langle wa,\ yb^*\rangle = \tau(y^*wab) = 0$$

since $y \perp [w\mathsf{A}_0]$. It then follows that $[w\mathsf{A}]_2 \ominus [y\mathsf{A}]_2 = \{0\}$, so $[w\mathsf{A}]_2 = [y\mathsf{A}]_2$. Hence, by (3.3)

$$[h\mathsf{A}]_2 = [u^*w\mathsf{A}]_2 = [u^*y\mathsf{A}]_2 = [|y|\mathsf{A}]_2 = H^2(\mathsf{A}).$$

We turn to the Riesz factorization. We first need to extend (1.3) to all indices.

Proposition 3.3 *Let $0 < p < q \le \infty$. Then*

$$H^p(\mathsf{A}) \cap L^q(\mathsf{M}) = H^q(\mathsf{A}) \quad \textit{and} \quad H_0^p(\mathsf{A}) \cap L^q(\mathsf{M}) = H_0^q(\mathsf{A}),$$

where $H_0^p(\mathsf{A}) = [\mathsf{A}_0]_p$.

Proof. It is obvious that $H^q(\mathsf{A}) \subset H^p(\mathsf{A}) \cap L^q(\mathsf{M})$. To prove the converse inclusion, we first consider the case $q = \infty$. Thus let $x \in H^p(\mathsf{A}) \cap \mathsf{M}$. Then by Corollary 2.2,

$$\forall\, a \in \mathsf{A}_0 \quad \Phi(xa) = \Phi(x)\Phi(a) = 0.$$

Hence by (1.1), $x \in \mathsf{A}$.

Now consider the general case. Fix an $x \in H^p(\mathsf{A}) \cap L^q(\mathsf{M})$. Applying Theorem 3.1 to $w = (x^*x+1)^{1/2}$, we get an invertible $h \in H^q(\mathsf{A})$ such that

$$h^*h = x^*x + 1 \quad \text{and} \quad h^{-1} \in \mathsf{A}.$$

Since $h^*h \le x^*x$, there exists a contraction $v \in \mathsf{M}$ such that $x = vh$. Then $v = xh^{-1} \in H^p(\mathsf{A}) \cap \mathsf{M}$, so $v \in \mathsf{A}$. Consequently, $x \in \mathsf{A} \cdot H^q(\mathsf{A}) = H^q(\mathsf{A})$. Thus we proved the first equality. The second is then an easy consequence. For this it suffices to note that $H_0^p(\mathsf{A}) = \{x \in H^p(\mathsf{A})\ :\ \Phi(x) = 0\}$. The later equality follows from the continuity of Φ on $H^p(\mathsf{A})$. $\square$

Theorem 3.4 *Let $0 < p, q, r \le \infty$ such that $1/p = 1/q + 1/r$. Then for $x \in H^p(\mathsf{A})$ and $\varepsilon > 0$ there exist $y \in H^q(\mathsf{A})$ and $z \in H^r(\mathsf{A})$ such that*

$$x = yz \quad \textit{and} \quad \|y\|_q\, \|z\|_r \le \|x\|_p + \varepsilon.$$

Consequently,

$$\|x\|_p = \inf\big\{\|y\|_q\, \|z\|_r\ :\ x = yz,\ y \in H^q(\mathsf{A}),\ z \in H^r(\mathsf{A})\big\}.$$

Proof. The case where $\max(q,\ r) = \infty$ is trivial. Thus we assume both q and r to be finite. Let $w = (x^*x + \varepsilon)^{1/2}$. Then $w \in L^p(\mathsf{M})$ and $w^{-1} \in \mathsf{M}$. Let $v \in \mathsf{M}$ be a contraction such that $x = vw$. Now applying Theorem 3.1 to $w^{p/r}$, we have: $w^{p/r} = uz$, where u is a unitary in M and $z \in H^r(\mathsf{A})$ such that $z^{-1} \in \mathsf{A}$. Set $y = vw^{p/q}u$. Then $x = yz$, so $y = xz^{-1}$. Since $x \in H^p(\mathsf{A})$ and $z^{-1} \in \mathsf{A}$, $y \in H^p(\mathsf{A})$. On the other hand, y belongs to $L^q(\mathsf{M})$ too. Therefore, $y \in H^q(\mathsf{A})$ by virtue of Proposition 3.3. The norm estimate is clear. $\square$

Remark 3.5 It is unknown at the time of this writing whether the infimum in Theorem 3.4 is attained. We will see in section 4 that the answer is affirmative if additionally $\Delta(x) > 0$.

4 Outer operators

We consider in this section outer operators. All results below on the left and right outers are due to Blecher and Labuschagne [2] in the case of indices not less than one. The notion of bilaterally outer is new. We start with the following result.

Proposition 4.1 *Let $0 < p < q \leq \infty$ and let $h \in H^q(\mathsf{A})$. Then*

i) $[h\mathsf{A}]_p = H^p(\mathsf{A})$ *iff* $[h\mathsf{A}]_q = H^q(\mathsf{A})$;

ii) $[\mathsf{A}h]_p = H^p(\mathsf{A})$ *iff* $[\mathsf{A}h]_q = H^q(\mathsf{A})$;

iii) $[\mathsf{A}h\mathsf{A}]_p = H^p(\mathsf{A})$ *iff* $[\mathsf{A}h\mathsf{A}]_q = H^q(\mathsf{A})$.

Proof. We prove only the third equivalence. The proofs of the two others are similar (and even simpler). It is clear that $[\mathsf{A}h\mathsf{A}]_q = H^q(\mathsf{A}) \ \Rightarrow\ [\mathsf{A}h\mathsf{A}]_p = H^p(\mathsf{A})$. To prove the converse implication we first consider the case $q \geq 1$. Let q' be the conjugate index of q. Let $x \in L^{q'}(\mathsf{M})$ be such that

$$\forall\, a, b \in \mathsf{A} \quad \tau(xahb) = 0.$$

Then $xah \in H^1_0(\mathsf{A})$ for any $a \in \mathsf{A}$ by virtue of (1.2) (more rigorously, its H^p_0-analogue as in Proposition 3.3). On the other hand, by the assumption that $[\mathsf{A}h\mathsf{A}]_p = H^p(\mathsf{A})$, there exist two sequences $(a_n), (b_n) \subset \mathsf{A}$ such that

$$\lim_n a_n h b_n = 1 \quad \text{in} \quad H^p(\mathsf{A}).$$

Consequently,

$$\lim_n x a_n h b_n = x \quad \text{in} \quad L^r(\mathsf{M}),$$

where $1/r = 1/q' + 1/p$. Since $xa_nhb_n = (xa_nh)b_n \in H^1_0(\mathsf{A}) \subset H^r_0(\mathsf{A})$, we deduce that $x \in H^r_0(\mathsf{A})$. Therefore, $x \in H^r_0(\mathsf{A}) \cap L^{q'}(\mathsf{M})$, so by Proposition 3.3, $x \in H^{q'}_0(\mathsf{A})$. Hence, $\tau(xy) = 0$ for all $y \in H^q(\mathsf{A})$. Thus $[\mathsf{A}hA]_q = H^q(\mathsf{A})$.

Now assume $q < 1$. Choose an integer n such that $np \geq 2$. By the proof of Theorem 3.4 and Remark 3.2, we deduce a factorization:

$$h = h_1\, h_2\, \cdots\, h_n\,,$$

where $h_k \in H^{np}(\mathsf{A})$ for every $1 \leq k \leq n$ and $[h_k\mathsf{A}]_2 = H^2(\mathsf{A})$ for $2 \leq k \leq n$. By the left version (i.e., part i)) of the previous case already proved, we also have $[h_k\mathsf{A}]_{np} = H^{np}(\mathsf{A})$ and $[h_k\mathsf{A}]_{nq} = H^{nq}(\mathsf{A})$ for $2 \leq k \leq n$. Let us deal with the first factor h_1. Using $[\mathsf{A}h\mathsf{A}]_p = H^p(\mathsf{A})$ and $[h_k\mathsf{A}]_{np} = H^{np}(\mathsf{A})$ for $2 \leq k \leq n$, we see that $[\mathsf{A}h_1\mathsf{A}]_p = H^p(\mathsf{A})$; so again $[\mathsf{A}h_1\mathsf{A}]_{nq} = H^{nq}(\mathsf{A})$ by virtue of the first part. It is then clear that $[\mathsf{A}h\mathsf{A}]_q = H^q(\mathsf{A})$. $\square$

The previous result justifies the relative independence of the index p in the following definition.

Definition 4.2 Let $0 < p \leq \infty$. An operator $h \in H^p(\mathsf{A})$ is called *left outer*, *right outer* or *bilaterally outer* according to $[h\mathsf{A}]_p = H^p(\mathsf{A})$, $[\mathsf{A}h]_p = H^p(\mathsf{A})$ or $[\mathsf{A}h\mathsf{A}]_p = H^p(\mathsf{A})$.

Remark 4.3 It is easy to see that if h is left outer or right outer, h is of full support (i.e., h is injective and of dense range). There exist, however, bilaterally outers which are not of full support. For example, consider the case where $\mathsf{A} = \mathsf{M} = \mathbb{M}_n$, the full algebra of $n \times n$ complex matrices, equipped with the normalized trace. Then every e_{ij} is bilaterally outer, where the e_{ij} are the canonical matrix units of $\mathbb{M}_n$. A less trivial case is the following. Let $\mathbb{T}$ be the unit circle equipped with normalized Haar measure. Let $\mathsf{M} = L^\infty(\mathbb{T})\bar{\otimes}\mathbb{M}_n = L^\infty(\mathbb{T};\mathbb{M}_n)$, and let $\mathsf{A} = H^\infty(\mathbb{T};\mathbb{M}_n)$, the algebra of $\mathbb{M}_n$-valued bounded analytic functions in the unit disc of the complex plane. Let $\varphi \in H^p(\mathbb{T})$ be an outer function. Then $h = \varphi \otimes e_{ij}$ is bilaterally outer with respect to A.

Theorem 4.4 *Let $0 < p \leq \infty$ and $h \in H^p(\mathsf{A})$.*

i) *If h is left or right outer, then $\Delta(h) = \Delta(\Phi(h))$. Conversely, if $\Delta(h) = \Delta(\Phi(h))$ and $\Delta(h) > 0$, then h is left and right outer (so bilaterally outer too).*

ii) *If A is antisymmetric (i.e., $\dim \mathsf{D} = 1$) and h is bilaterally outer, then $\Delta(h) = \Delta(\Phi(h))$.*

Proof. i) This part is proved in [2] for $p \geq 1$. Assume h is left outer. Let $d \in \mathsf{D}$. Using Theorem 2.1, we obtain

$$\|\Phi(h)d\|_p = \inf\big\{\|hd + x_0\|_p \;:\; x \in H^p_0(\mathsf{A})\big\}.$$

On the other hand,

$$[h\mathsf{A}_0]_p = \big[[h\mathsf{A}]_p\mathsf{A}_0\big]_p = \big[[\mathsf{A}]_p\mathsf{A}_0\big]_p = [\mathsf{A}_0]_p = H^p_0(\mathsf{A}).$$

Therefore,

$$\|\Phi(h)d\|_p = \inf\big\{\|h(d + a_0)\|_p \;:\; a_0 \in \mathsf{A}_0\big\}.$$

Recall the following characterization of $\Delta(x)$ from [2]:

$$\Delta(x) = \inf\big\{\|xa\|_p \;:\; a \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1\big\}. \tag{4.1}$$

Now using this formula twice, we obtain

$$\begin{aligned}\Delta(\Phi(h)) &= \inf\big\{\|\Phi(h)d\|_p \;:\; d \in \mathsf{D}, \Delta(d) \geq 1\big\}\\ &= \inf\big\{\|h(d + a_0)\|_p \;:\; d \in \mathsf{D}, \Delta(d) \geq 1,\ a_0 \in \mathsf{A}_0\big\}\\ &= \Delta(h).\end{aligned}$$

Let us show the converse under the additional assumption that $\Delta(h) > 0$. We will use the case $p \geq 1$ already proved in [2]. Thus assume $p < 1$. Choose an integer n such that $np \geq 1$. By Theorem 3.4, there exist $h_1, \dots, h_n \in H^{np}(\mathsf{A})$ such that $h = h_1 \cdots h_n$. Then $\Delta(h) = \Delta(h_1) \cdots \Delta(h_n)$; so $\Delta(h_k) > 0$ for all $1 \leq k \leq n$. On the other hand, by Arveson-Labuschagne's Jensen inequality [1, 11] (or Corollary 2.3), $\Delta(\Phi(h_k)) \leq \Delta(h_k)$. However,

$$\Delta(\Phi(h)) = \Delta(\Phi(h_1)) \cdots \Delta(\Phi(h_n)) \leq \Delta(h_1) \cdots \Delta(h_n) = \Delta(h) = \Delta(\Phi(h)).$$

It then follows that $\Delta(\Phi(h_k)) = \Delta(h_k)$ for all k. Now $h_k \in H^{np}(\mathsf{A})$ with $np \geq 1$, so h_k is left and right outer. Consequently, h is left and right outer.

ii) This proof is similar to that of the first part of i). We will use the following variant of (4.1)

$$\Delta(x) = \inf\big\{\|axb\|_p \;:\; a, b \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1,\ \Delta(\Phi(b)) \geq 1\big\} \tag{4.2}$$

for every $x \in L^p(\mathsf{M})$. This formula immediately follows from (4.1). Indeed, by (4.1) and the multiplicativity of Δ

$$\begin{aligned}&\inf\big\{\|axb\|_p \ : \ a, b \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1,\ \Delta(\Phi(b)) \geq 1\big\}\\ &= \inf\big\{\Delta(ax) \ : \ a \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1\big\}\\ &= \inf\big\{\Delta(a)\Delta(x) \ : \ a \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1\big\} = \Delta(x).\end{aligned}$$

Now assume $h \in H^p(\mathsf{A})$ is bilaterally outer and A is antisymmetric. Then $\Phi(h)$ is a multiple of the unit of M. As in the proof of i), We have

$$\|\Phi(h)\|_p = \inf\big\{\|h + x\|_p \ : \ x \in H^p_0(\mathsf{A})\big\} = \inf\big\{\|h + ahb_0\|_p \ : \ a \in \mathsf{A},\ b_0 \in \mathsf{A}_0\big\}. \tag{4.3}$$

Using $\dim \mathsf{D} = 1$, we easily check that

$$\inf\big\{\|h + ahb_0\|_p \ : \ a \in \mathsf{A},\ b_0 \in \mathsf{A}_0\big\} = \inf\big\{\|(1 + a_0)h(1 + b_0)\|_p \ : \ a_0, b_0 \in \mathsf{A}_0\big\}. \tag{4.4}$$

Indeed, it suffices to show that both sets $\{h + ahb_0 : a \in \mathsf{A},\ b_0 \in \mathsf{A}_0\}$ and $\{(1 + a_0)h(1 + b_0) : a_0, b_0 \in \mathsf{A}_0\}$ are dense in $\{x \in H^p(\mathsf{A}) : \Phi(x) = \Phi(h)\}$. The first density immediately follows from the density of $\mathsf{A}h\mathsf{A}_0$ in $H^p_0(\mathsf{A})$. On the other hand, let $x \in H^p(\mathsf{A})$ with $\Phi(x) = \Phi(h)$ and let $a_n, b_n \in \mathsf{A}$ such that $\lim_n a_nhb_n = x$. By Theorem 2.1,

$$\lim_n \Phi(a_n)\Phi(h)\Phi(b_n) = \Phi(x).$$

Since $\Phi(x) = \tau(x)1 = \tau(h)1 = \Phi(h) \neq 0$, we deduce that $\lim_n \tau(a_n)\tau(b_n) = 1$. Thus replacing a_n and b_n by $a_n/\tau(a_n)$ and $b_n/\tau(b_n)$, respectively, we can assume that $a_n = 1 + \widetilde{a}_n$ and $b_n = 1 + \widetilde{b}_n$ with $\widetilde{a}_n, \widetilde{b}_n \in \mathsf{A}_0$; whence the desired density of $\{(1 + a_0)h(1 + b_0) : a_0, b_0 \in \mathsf{A}_0\}$ in $\{x \in H^p(\mathsf{A}) : \Phi(x) = \Phi(h)\}$. Finally, combining (4.2), (4.3) and (4.4), we get $\Delta(\Phi(h)) = \Delta(h)$. $\square$

Remark 4.5 The assumption that A is antisymmetric in Theorem 4.4, ii) cannot be removed in general, as shown by the following example. Keep the notation introduced in Remark 4.3 and consider the case where $\mathsf{M} = L^\infty(\mathbb{T}; \mathbb{M}_2)$ and $\mathsf{A} = H^\infty(\mathbb{T}; \mathbb{M}_2)$. Let φ_1 and φ_2 be two outer functions in $H^p(\mathbb{T})$, and let $h = \varphi_1 \otimes e_{11} + z\varphi_2 \otimes e_{22}$, where z denotes the identity function on $\mathbb{T}$. Then it is easy to check that h is bilaterally outer and

$$\Delta(h) = \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \log|\varphi_1| + \frac{1}{2}\int_{\mathbb{T}} \log|\varphi_2|\Big) > 0.$$

However, $\Phi(h) = \varphi_1(0)e_{11}$, so $\Delta(\Phi(h)) = 0$.

The following is an immediate consequence of Theorem 4.4. We do not know, however, whether the condition $\Delta(h) > 0$ in i) can be removed or not.

Corollary 4.6 *Let $h \in H^p(\mathsf{A})$, $0 < p \leq \infty$.*

i) *If $\Delta(h) > 0$, then h is left outer iff h is right outer.*

ii) *Assume that A is antisymmetric. Then the following properties are equivalent:*

- *h is left outer;*
- *h is right outer;*
- *h is bilaterally outer;*
- *$\Delta(\Phi(h)) = \Delta(h) > 0$.*

We will say that h is *outer* if it is at the same time left and right outer. Thus if $h \in H^p(\mathsf{A})$ with $\Delta(h) > 0$, then h is outer iff $\Delta(h) = \Delta(\Phi(h))$. Also in the case where A is antisymmetric, an h with $\Delta(h) > 0$ is outer iff it is left, right or bilaterally outer.

Corollary 4.7 *Let $h \in H^p(\mathsf{A})$ such that $h^{-1} \in H^q(\mathsf{A})$ with $0 < p, q \leq \infty$. Then h is outer.*

Proof. By the multiplicativity of Δ, $\Delta(h)\Delta(h^{-1}) = 1$ and $\Delta(\Phi(h))\Delta(\Phi(h^{-1})) = 1$. Thus by Jensen's inequality (Corollary 2.3),

$$\Delta(h) = \Delta(h^{-1})^{-1} \leq \Delta(\Phi(h^{-1}))^{-1} = \Delta(\Phi(h));$$

whence the assertion because of Theorem 4.4. $\square$

The following improves Theorem 3.1.

Theorem 4.8 *Let $w \in L^p(\mathsf{M})$ with $0 < p \leq \infty$ such that $\Delta(w) > 0$. Then there exist a unitary $u \in \mathsf{M}$ and an outer $h \in H^p(\mathsf{A})$ such that $w = uh$.*

Proof. Based on the case $p \geq 1$ from [2], the proof below is similar to the end of the proof of Theorem 3.1. For simplicity we consider only the case where $p \geq 1/2$. Write the polar decomposition of w: $w = v|w|$. Applying [2] to $|w|^{1/2}$ we get a factorization: $|w|^{1/2} = u_2 h_2$ with u_2 unitary and $h_2 \in H^{2p}(\mathsf{A})$ left outer. Since $\Delta(h_2) > 0$, h_2 is also right outer; so h_2 is outer. Similarly, we have: $v|w|^{1/2}u_2 = u_1 h_1$. Then $u = u_1$ and $h = h_1 h_2$ yield the desired factorization of w. $\square$

The following is the inner-outer factorization for operators in $H^p(\mathsf{A})$, which is already in [2] for $p \geq 1$.

Corollary 4.9 *Let $0 < p \leq \infty$ and $x \in H^p(\mathsf{A})$ with $\Delta(x) > 0$. Then there exist a unitary $u \in \mathsf{A}$ (inner) and an outer $h \in H^p(\mathsf{A})$ such that $x = uh$.*

Proof. Applying the previous theorem, we get $x = uh$ with h outer and u a unitary in M. Let $a_n \in \mathsf{A}$ such that $\lim ha_n = 1$ in $H^p(\mathsf{A})$. Then $u = \lim xa_n$ in $H^p(\mathsf{A})$ too; so $u \in H^p(\mathsf{A}) \cap \mathsf{M}$. By Proposition 3.3, $u \in \mathsf{A}$. $\square$

Remark 4.10 The condition $\Delta(x) > 0$ cannot be removed in general. Indeed, if h is outer, then h is of full support (see Remark 4.5). It follows that x is of full support too if x admits an inner-outer factorization as above. Consider, for instance, the example in Remark 4.5. Then for any $\varphi \in H^p(\mathbb{T})$ the operator $x = \varphi \otimes e_{11} \in H^p(\mathsf{A})$ is not of full support.

Corollary 4.11 *Let $0 < p \leq \infty$ and $h \in H^p(\mathsf{A})$ with $\Delta(h) > 0$. Then h is outer iff for any $x \in H^p(\mathsf{A})$ with $|x| = |h|$ we have $\Delta(\Phi(x)) \leq \Delta(\Phi(h))$.*

Proof. Assume h outer. Then by Corollary 2.3 and Theorem 4.4,

$$\Delta(\Phi(x)) \leq \Delta(x) = \Delta(h) = \Delta(\Phi(h)).$$

Conversely, let $h = uk$ be the decomposition given by Theorem 4.8 with k outer. Then

$$\Delta(h) = \Delta(k) = \Delta(\Phi(k)) \leq \Delta(\Phi(h));$$

so h is outer by Theorem 4.4. $\square$

Corollary 4.12 *Let $0 < p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$. Let $x \in H^p(\mathsf{A})$ be such that $\Delta(x) > 0$. Then there exist $y \in H^q(\mathsf{A})$ and $z \in H^r(\mathsf{A})$ such that*

$$x = yz \quad \text{and} \quad \|x\|_p = \|y\|_q \, \|z\|_r \, .$$

Proof. This proof is similar to that of Theorem 3.4. Instead of Theorem 3.1, we now use Theorem 4.8. Indeed, by the later theorem, we can find a unitary $u_2 \in \mathsf{M}$ and an outer $h_2 \in H^{p/r}(\mathsf{A})$ such that $|x|^{p/r} = u_2 h_2$. Once more applying this theorem to $v|x|^{p/q}\, u_2$, we have a similar factorization: $v|x|^{p/q}\, u_2 = u_1 h_1$, where v is the unitary in the polar decomposition of x. Since h_1 and h_2 are outer, we deduce, as in the proof of Corollary 4.9, that $u_1 \in \mathsf{A}$. Then $y = u_1 h_1$ and $z = h_2$ give the desired factorization of x. $\square$

5 A noncommutative Szegö formula

Let $w \in L^1(\mathbb{T})$ be a positive function and let $d\mu = wdm$. Then we have the following well-known Szegö formula [16]:

$$\inf\Big\{\int_{\mathbb{T}} |1-f|^2 d\mu \ :\ f \text{ mean zero analytic polynomial}\Big\} = \exp\int_{\mathbb{T}} \log w dm.$$

This formula was later proved for any positive measure μ on $\mathbb{T}$ independently by Kolmogorov/Krein [10] and Verblunsky [18]. Then the singular part of μ with respect to the Lebesgue measure dm does not contribute to the preceding infimum and w on the right hand side is the density of the absolute part of μ (also see [8]). This latter result was extended to the noncommutative setting in [2]. More precisely, let ω be a positive linear functional on M, and let $\omega = \omega_n + \omega_s$ be the decomposition of ω into its normal and singular parts. Let $w \in L^1(\mathsf{M})$ be the density of ω_n with respect to τ, i.e., $\omega_n = \tau(w\,\cdot)$. Then Blecher and Labuschagne proved that if $\dim \mathsf{D} < \infty$,

$$\Delta(w) = \inf\big\{\omega(|a|^2) \ :\ a \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1\big\}.$$

It is left open in [2] whether the condition $\dim \mathsf{D} < \infty$ can be removed or not. We will solve this problem in the affirmative. At the same time, we show that the square in the above formula can be replaced by any power p.

Theorem 5.1 *Let $\omega = \omega_n + \omega_s$ be as above and $0 < p < \infty$. Then*

$$\Delta(w) = \inf\big\{\omega(|a|^p) \ :\ a \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1\big\}.$$

Proof. Let

$$\delta(\omega) = \inf\{\omega(|a|^p) \ :\ a \in \mathsf{A},\ \Delta(\Phi(a)) \geq 1\}.$$

First we show that

$$\delta(\omega) = \inf\{\omega(x) \ :\ x \in \mathsf{M}_+^{-1},\ \Delta(x) \geq 1\},$$

where M_+^{-1} denotes the family of invertible positive operators in M with bounded inverses. Given any $x \in \mathsf{M}_+^{-1}$, by Arveson's factorization theorem there exists $a \in \mathsf{A}$ such that $|a| = x^{1/p}$ and $a^{-1} \in \mathsf{A}$. Then $x = |a|^p$, so $\Delta(x) = \Delta(|a|^p) = \Delta(a)^p$. Since a is invertible with $a^{-1} \in \mathsf{A}$, by Jensen's formula in [1], $\Delta(a) = \Delta(\Phi(a))$. It then follows that

$$\delta(\omega) \leq \inf\{\omega(x) \ :\ x \in \mathsf{M}_+^{-1},\ \Delta(x) \geq 1\}.$$

The converse inequality is easier. Indeed, given $a \in \mathsf{A}$ with $\Delta(\Phi(a)) \geq 1$ and $\varepsilon > 0$, set $x = |a|^p + \varepsilon$. Then $x \in \mathsf{M}_+^{-1}$ and $\Delta(x) \geq \Delta(a)^p \geq \Delta(\Phi(a))^p$ by virtue of Jensen's inequality. Since $\lim_{\varepsilon\to 0} \omega(|a|^p + \varepsilon) = \omega(|a|^p)$, we deduce the desired converse inequality.

Next we show that $\delta(\omega) = \delta(\omega_n)$. The singularity of ω_s implies that there exists an increasing net (e_i) of projections in M such that $e_i \to 1$ strongly and $\omega_s(e_i) = 0$ for every i (see [17, III.3.8]). Let $\varepsilon > 0$. Set

$$x_i = \varepsilon^{\tau(e_i)-1}(e_i + \varepsilon e_i^\perp), \quad \text{where} \quad e^\perp = 1 - e.$$

Clearly, $x_i \in \mathsf{M}_+^{-1}$ and $\Delta(x_i) = 1$. Let $x \in \mathsf{M}_+^{-1}$ and $\Delta(x) \geq 1$. Then $\Delta(x_i x x_i) = \Delta(x) \geq 1$, and $x_i x x_i \to x$ in the w*-topology. On the other hand, note that

$$\omega_s(x_i x x_i) = \varepsilon^{2\tau(e_i)}\omega_s(e_i^\perp x e_i^\perp).$$

Therefore,

$$\begin{aligned}
\delta(\omega) &\leq \limsup \omega(x_i x x_i) = \omega_n(x) + \limsup \omega_s(x_i x x_i)\\
&\leq \omega_n(x) + \limsup \varepsilon^{2\tau(e_i)}\omega_s(e_i^\perp x e_i^\perp)\\
&\leq \omega_n(x) + \varepsilon^2\|\omega_s\|\,\|x\|.
\end{aligned}$$

It thus follows that $\delta(\omega) \leq \delta(\omega_n)$, so $\delta(\omega) = \delta(\omega_n)$. Now it is easy to conclude the validity of the result. Indeed, the preceding two parts imply

$$\delta(\omega) = \inf\{\tau(wx) \ : \ x \in \mathsf{M}_+^{-1},\ \Delta(x) \geq 1\}.$$

By a formula on determinants from [1], the last infimum is nothing but $\Delta(w)$. Therefore, the theorem is proved. $\square$

Remark 5.2 The proof above shows that the infimum in Theorem 5.1 remains the same if one requires a to be invertible with $a^{-1} \in \mathsf{A}$ (i.e., $a \in \mathsf{A}^{-1}$). Namely,

$$\delta(\omega) = \inf\{\omega(|a|^p) \ : \ a \in \mathsf{A}^{-1},\ \Delta(\Phi(a)) \geq 1\} = \inf\{\omega(|a|^p) \ : \ a \in \mathsf{A}^{-1},\ \Delta(a) \geq 1\}.$$

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